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THE METHOD OF LYAPUNOV FUNCTIONS IN PROBLEMS OF MULTISTABILITY OF MOTION*

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The multistability of motion is defined as the property whereby different groups of variables describing the motion have different types of stability, e.g., one group has stability in the small, another has asymptotic stability, and another has boundedness etc. The method of Lyapunov functions is used to prove theorems on multistability which are then used to study the stability of motion of winged aircraft with respect to groups of variables. In existing definitions and studies of stability it is usually assumed that the phase coordinates have the same type of stability, e.g., asymptotic or uniform, etc. In practice, however, say when synthesizing aircraft trajectories, we need to take account of different requirements imposed on the behaviour of different groups of phase coordinates. For instance, when considering the space manoeuvres of an aircraft with constant load factor, it is important to obtain asymptotic stability with respect to the angles of attack and sideslip, while only uniform stability with respect to the angles of pitch, yaw, and rotation is needed. The angles of pitch, yaw, and rotation can themselves have any values, i.e., their stable behaviour is not required. In short, the individual coordinates or groups of coordinates of the same system can have different types of stability, say asymptotic or uniform. We then speak of the multistable motion of the system. It is a further development of the idea of partial stability /1/, the theory of which has been taken further by Rumyantsev /2/ and his associates, and by others /3/.

1. Consider the equation of perturbed motion

$$d\mathbf{x}/dt = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, 0) \equiv 0, \quad t \geqslant 0 \qquad (1.1)$$
$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n, \quad t \in R^1$$

Corresponding to the unperturbed motion we have $\mathbf{x} = \mathbf{x}(t) \equiv 0$.
We divide the phase coordinates $x_1, x_2, \ldots, x_n$ into N groups:

$$\mathbf{x}^{(1)} = (x_1, x_2, \ldots, x_{n_1}), \quad \mathbf{x}^{(2)} = (x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2}), \ldots \qquad (1.2)$$
$$\ldots, \mathbf{x}^{(j)} = (x_{m_{j-1}+1}, x_{m_{j-1}+2}, \ldots, x_{m_{j-1}+n_j}), \ldots, \mathbf{x}^{(N)} =$$
$$(x_{m_{N-1}+1}, x_{m_{N-1}+2}, \ldots, x_n); \quad m_{j-1} = \sum_{k=1}^{j-1} n_k, \quad n = \sum_{k=1}^{N} n_k.$$
$$j = 1, 2, \ldots, N$$

We introduce the notation and norms

$$x_i^{(j)} = x_i, \quad i = m_{j-1}+1, \quad m_{j-1}+2, \ldots, m_{j-1}+n_j; \quad m_0 \equiv 0 \qquad (1.3)$$
$$\mathbf{x} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_{n_1}^{(1)}, x_{n_1+1}^{(2)}, \ldots, x_{n_1+n_2}^{(2)}, x_{n_1+n_2+1}^{(3)}, \ldots, x_n^{(N)})$$
$$\|\mathbf{x}^{(j)}\| = \Big(\sum_{i=1}^{n_j} x_{m_{j-1}+i}^2\Big)^{1/2}, \quad \|\mathbf{x}^{(j,k)}\| =$$
$$(\|\mathbf{x}^{(j)}\|^2 + \|\mathbf{x}^{(j+1)}\|^2 + \ldots + \|\mathbf{x}^{(k)}\|^2)^{1/2}, \quad j < k$$
$$\|\mathbf{x}\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} = \Big(\sum_{j=1}^{N} \|\mathbf{x}^{(j)}\|^2\Big)^{1/2}, \quad j = 1, 2, \ldots, N$$

Similar notation and norms are introduced for the components of the vector $\mathbf{X}$ (1.1), i.e., for $\mathbf{X}^{(j)}$, $\mathbf{X}^{(j,k)}$, $j < k$, $j = 1, 2, \ldots, N$.

Let $\mathbf{x} = \mathbf{x}(t;\, t_0, \mathbf{x}_0)$ be the solution of system (1.1) with the initial data t_0, $\mathbf{x}_0 = \mathbf{x}(t_0;\, t_0, \mathbf{x}_0)$.

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We shall consider the multistability of the solution $x \equiv 0$ with respect to all variables $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$ or with respect to some of the variables $x^{(1)}, x^{(2)}, \ldots, x^{(N_*)}$, where $N_* < N$. We therefore assume that the right-hand sides of (1.1) are continuous functions in the domain

$$G^{(N_*)} = \{t, x: \| x^{(1, N_*)} \| \leqslant H > 0, \; 0 \leqslant \| x^{(N_*+1, N)} \| < \infty, t \in [0, \infty)\} \tag{1.4}$$

and that they satisfy the conditions for the solution $x = x(t; t_0, x_0)$ to be unique, which is defined for all $t \geqslant 0$ with $\| x^{(1, N_*)} \| \leqslant H$, i.e., we have $x^{(1, N_*)}$ continuability of the solution /3/. If we consider the multistability of the solution $x \equiv 0$ with respect to all variables $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$, we put $N_* = N$ and our assumptions hold in the domain $G^{(N)}$.

Definition 1. We call the solution $x \equiv 0$ of system (1.1) multistable when

1) it is $x^{(1)}$-stable, i.e., if, for any $\varepsilon > 0$, $t_0 \geqslant 0$, no matter how small ε is, there exists $\delta(\varepsilon, t_0) > 0$ such that $\| x_0 \| < \delta$ implies $\| x^{(1)}(t; t_0, x_0) \| < \varepsilon$, $\forall t \geqslant t_0$;

2) it is $x^{(2)}$-stable uniformly with respect to t_0, i.e., if it is $x^{(2)}$-stable and for any $\varepsilon > 0$ we can choose $\delta(\varepsilon)$, independent of t_0;

3) it is asymptotically $x^{(3)}$-stable, i.e., if it is $x^{(3)}$-stable and for any $t_0 \geqslant 0$ there exists $\Delta(t_0) > 0$ such that the solution $x(t; t_0, x_0)$ with initial value $\| x_0 \| < \Delta$ satisfies the condition

$$\lim_{t \to \infty} \| x^{(3)}(t; t_0, x_0) \| = 0$$

4) is asymptotically $x^{(4)}$-stable uniformly with respect to $\{t_0, x_0\}$, i.e., if it is $x^{(4)}$-stable uniformly with respect to t_0 and there exists a number $\Delta_0 > 0$, independent of t_0, such that the condition

$$\lim_{t \to \infty} \| x^{(4)}(t; t_0, x_0) \| = 0$$

holds uniformly with respect to $\{t_0, x_0\}$ of the domain $G^\circ = \{t_0, x_0: t_0 \geqslant 0, \; \| x_0 \| < \Delta_0\}$, i.e., for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that $t_0 \geqslant 0$, $\| x_0 \| < \Delta_0$ imply $\| x^{(4)}(t; t_0, x_0) \| < \varepsilon$ for all $t \geqslant t_0 + T$;

5) every j-th group of variables $(j = 1, 2, \ldots, N)$ has a definite type of stability, where $n_1 + n_2 + \ldots + n_{N_*} = n$, $N_* = N$.

With $n_1 + n_2 + \ldots + n_{N_*} < n$, i.e., $N_* < N$ the behaviour of the group of variables with subscripts $n_{N_*+1}, n_{N_*+2}, \ldots, n_N$ is not controlled and we arrive at the concept of multistability with respect to some of the variables $x^{(1)}, x^{(2)}, \ldots, x^{(N_*)}$.

We consider the real functions $V(t, x)$ which are defined and continuous in the domain $G^{(N_*)}$, have continuous partial derivatives $\partial V/\partial t$, $\partial V/\partial x_i$ $(i = 1, 2, \ldots, n)$ at all points of this domain, and satisfy the condition $V(t, 0) \equiv 0$.

Definition 2 /1/. The function $W(x^{(1, N_*)})$, not explicitly dependent on time t, is called positive definite with respect to the variables $x^{(1, N_*)}$ if it is non-negative in the domain $\| x^{(1,N_*)} \| \leqslant H$ and vanishes if and only if $x^{(1,N_*)} = 0$.

Definition 3 /2/. The function $V(t, x)$ is called $x^{(1,N_*)}$ positive definite if there exists a positive definite function $W(x^{(1,N_*)})$, not explicitly dependent on t, such that, in the domain $G^{(N_*)}$ given by (1.4), we have

$$V(t, x) \geqslant W(x^{(1,N_*)}) \tag{1.5}$$

Lemma 1. The necessary and sufficient condition for $V(t, x)$ to be $x^{(1,N_*)}$ positive definite is that it can be written as the sum of a non-negative function $V_+(t, x)$ with respect to all variables $x_1, x_2, \ldots, x_n$ and a function $W(x^{(1,N_*)})$, not explicitly dependent on t, which is positive definite with respect to the variables $x^{(1,N_*)}$, i.e.,

$$V(t, x) = V_+(t, x) + W(x^{(1,N_*)}) \tag{1.6}$$

Proof. Necessity. Let $V(t, x)$ be $x^{(1,N_*)}$ positive definite. Then, by Definition 3, there is a positive definite function $W(x^{(1,N_*)})$, such that, in the domain $G^{(N_*)}$ of (1.4), condition (1.5) holds.

We introduce the function

$$V_+(t, x) = V(t, x) - W(x^{(1,N_*)}) \tag{1.7}$$

which, by condition (1.5), is non-negative. From (1.7) we obtain expression (1.6) for $V(t, x)$.

Sufficiency. Let Eq.(1.6) hold, where $V_+(t, x) \geqslant 0$, and $W(x^{(1,N_*)})$ is positive definite with respect to the variable $x^{(1,N_*)}$. Then it follows from (1.6) that $V(t, x) - W(x^{(1,N_*)}) = V_+(t, x) \geqslant 0$. Hence condition (1.5) hold for $V(t, x)$, i.e., $V(t, x)$ is $x^{(1,N_*)}$ positive definite.

Lemma 2 /3/. The function $V(t, x)$ is $x^{(1,N*)}$ positive definite if and only if there is a continuous function $f(r)$, monotonically increasing with respect to $r \in [0, H] \subset R^1$, $f(0) = 0$, such that, in the domain $G^{(N*)}$ of (1.4),

$$V(t, x) \geqslant f(\| x^{(1,N*)} \|) \tag{1.8}$$

Definition 4 /3/. The function $V(t, x)$ admits of an infinitely small upper limit with respect to $x^{(1,N*)}$ if, given any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that $t \geqslant 0$, $\| x^{(1,N*)} \| < \delta$, $0 \leqslant \| x^{(N*+1,N)} \| < \infty$ implies $| V(t, x) | < \varepsilon$.

Definition 5 /4/. The function $V(t, x)$, defined in the domain

$$G = \{t, x\colon 0 \leqslant \| x \| < \infty, \quad t \in [0, \infty)\} \tag{1.9}$$

admits of an infinitely large lower limit with respect to $x^{(1,N*)}$ in G, if, in the domain G of (1.9), we have condition (1.5) and

$$W(x^{(1,N*)}) \to \infty \quad \text{as} \quad \| x^{(1,N*)} \| \to \infty \tag{1.10}$$

2. Let us prove some theorems on the multistability of motion. To be specific, we will divide the variables $x_1, x_2, \ldots, x_n$ into four groups, i.e., we take $N = 4$ and consider the multistability of the solution $x = x^{(1,4)} \equiv 0$ with respect to some of the variables $x^{(1,3)}$.

Definition 6. We call the solution $x = x^{(1,4)} \equiv 0$ of system (1.1) $x^{(1,3)}$-stable uniformly with respect to t_0, asymptotically $x^{(2,3)}$-stable, or $x^{(3)}$-stable in the large, if:

for any $\varepsilon > 0$, $t_0 \geqslant 0$, there is a number $\delta = \delta(\varepsilon) > 0$ such that $\| x^{(1,3)} \| < \delta$ $(0 \leqslant \| x^{(4)} \| < \infty)$ implies

$$\| x^{(1,3)}(t; t_0, x_0) \| < \varepsilon, \ \forall t \geqslant t_0$$

for any $t_0 \geqslant 0$ there exists $\Delta(t_0) > 0$ such that the solution $x(t; t_0, x_0)$ with $\| x_0 \| < \Delta$ has the property

$$\lim_{t\to\infty} \| x^{(2,3)}(t; t_0, x_0) \| = 0 \tag{2.1}$$

where $\| x \| < \Delta$ is the estimate of the domain of $x^{(2,3)}$-attraction of the point $x = 0$ for the initial instant t_0;

given μ_0 there exists $\mu = \mu(\mu_0) > 0$ such that, for any $\{x_0, t_0\}$ satisfying the inequality $\| x_0 \| < \mu_0$, $\forall t \geqslant 0$, we have

$$\| x^{(3)}(t; t_0, x_0) \| < \mu, \ \forall t \geqslant t_0$$

Theorem 1. For $x^{(1,3)}$-stability, uniform with respect to t_0, $x^{(2,3)}$-asymptotic stability, or $x^{(3)}$-stability in the large of the solution $x = x^{(1,4)} \equiv 0$ of system (1.1), it is sufficient that there exists in the domain $G^{(3)}$ of (1.4) a positive definite function $V(t, x)$ which, with respect to $x^{(1,3)}$, admits of an infinitely small upper limit and whose total time derivative, taken with the opposite sign, i.e., $-dV/dt$, is an $x^{(2,3)}$-positive definite function, and we have the conditions

$$\sup_{\|x\|<\mu_0} V(t, x) < \inf_{\|x^{(3)}\|=\mu} V(t, x), \quad \forall t \geqslant t_0 \tag{2.2}$$

$$\| X^{(2,3)} \| \leqslant M > 0, \quad M \in R^1 \tag{2.3}$$

Proof. Since $-dV/dt$ is an $x^{(2,3)}$-positive definite function, we have by Lemma 1:

$$-dV/dt = W_1(t, x) + W_2(x^{(2,3)}) \tag{2.4}$$

where $W_1(t, x) \geqslant 0$, $W_2(x^{(2,3)})$ is a positive definite function.

By (2.4) we have $dV/dt \leqslant 0$, i.e., Theorem 1 embraces the conditions of the theorem on t_0-uniform stability with respect to some of the variables /3/. Hence the solution $x \equiv 0$ of system (1.1) is stable uniformly with respect to t_0. Hence it follows that, given any $\varepsilon > 0$, $t_0 \geqslant 0$ there exists $\delta(\varepsilon)$ such that $\| x_0 \| < \delta$ implies $\| x^{(2,3)}(t; t_0, x_0) \| < \varepsilon$ for $t \geqslant t_0$.

Let us prove property (2.1). Assume the contrary: let there be a point x_* with $\| x_* \| < \delta$ $(\delta > 0)$, a number $l > 0)$, and a sequence $t_k \to \infty$, $t_k - t_{k-1} \geqslant \alpha > 0$, $k = 1, 2, \ldots$, such that $\| x^{(2,3)}(t_k; t_0, x_*) \| \geqslant l$. By (2.3), we can /5/ choose β, $0 < \beta < \alpha/2$, for which

$$l/2 \leqslant \| x^{(2,3)}(t; t_0, x_*) \| < \varepsilon, \ \forall t \in [t_k - \beta, t_k + \beta], \quad k = 1, 2, \ldots \tag{2.5}$$

By Lemma 2, for the $x^{(2,3)}$-positive definite function $(-dV/dt)$ we have

$$dV/dt \leqslant -f(\| x^{(2,3)} \|),$$

where the function $f(r)$ is continuous and monotonically increasing with respect to $r \in [0, H]$. On integrating this inequality between the limits $t_0 = t_1 - \beta$ and $t = t_k + \beta$ and using (2.5), we obtain

$$0 \leqslant V(t_k+\beta, \mathrm{x}(t_k+\beta; t_0, \mathrm{x}_*) \leqslant V(t_0, \mathrm{x}_*) - \int_{t_0}^{t} f(\|\mathrm{x}^{(2,3)}\|)\, dt \leqslant$$

$$V(t_0, \mathrm{x}_*) - \sum_{i=1}^{k} \int_{t_i-\beta}^{t_i+\beta} f(\|\mathrm{x}^{(1,3)}\|)\, dt \leqslant V(t_0, \mathrm{x}_*) - 2k\beta f\left(\frac{l}{2}\right).$$

The condition $V(t_k+\beta, \mathrm{x}(t_k+\beta; t_0, \mathrm{x}_*)) \geqslant 0$ is violated for sufficiently large k. Hence the assumption that $l > 0$ is impossible, i.e., $l = 0$ and condition (2.1) holds. The $\mathrm{x}^{(2,3)}$-asymptotic stability of the solution $\mathrm{x} \equiv 0$ of system (1.1) is thus proved.

To prove the stability of the solution $\mathrm{x} \equiv 0$ in the large with respect to the group of variables $\mathrm{x}^{(3)}$, we have to show that, under the conditions of the theorem, the norm $\|\mathrm{x}^{(3)}(t; t_0, x_0)\|$ does not reach a value equal to μ if at the initial instant $t = t_0$ we have $\|\mathrm{x}_0\| < \mu_0$.

Let $\|\mathrm{x}_0\| < \mu_0$. In the domain $\Gamma_\mu = \{\mathrm{x}, t\colon \|\mathrm{x}^{(3)}\| < \mu,\ t \geqslant t_0\}$ we have $dV/dt \leqslant 0$. Then,

$$V(t, \mathrm{x}) \leqslant V(t_0, \mathrm{x}_0) \leqslant \sup_{\|\mathrm{x}\|<\mu_0} V(t, \mathrm{x}) \tag{2.6}$$

Let us show that, under condition (2.2),

$$\|\mathrm{x}^{(3)}(t; t_0, \mathrm{x}_0)\| < \mu, \quad \forall t \geqslant t_0 \tag{2.7}$$

If this is not the case, i.e., the left-hand side of (2.7) is equal to μ at some instant $t = t_* > t_0$, then

$$V(t_*, \mathrm{x}) \geqslant \inf_{\|\mathrm{x}^{(3)}\|=\mu} V(t_*, \mathrm{x}) \tag{2.8}$$

Using (2.6) and (2.8), we have

$$\sup_{\|\mathrm{x}\|\leqslant\mu_0} V(t_*, \mathrm{x}) \geqslant V(t_*, \mathrm{x}) \geqslant \inf_{\|\mathrm{x}^{(3)}\|=\mu} V(t_*, \mathrm{x})$$

which contradicts condition (2.2) of the theorem. This contradiction proves the stability in the large.

Thus, all the properties of multistability of the solution, and hence Theorem 1, are proved.

In the next theorem we take $N = 3$ and assume that the conditions on the right-hand sides of system (1.1) hold in the domain G of (1.9).

Definition 7. The solution $\mathrm{x} = \mathrm{x}^{(1,3)} \equiv 0$ of system (1.1) is called $\mathrm{x}^{(1,2)}$-stable uniformly with respect to t_0, or asymptotically $\mathrm{x}^{(2)}$-stable in the large, if:

for any $\varepsilon > 0$, $t_0 \geqslant 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\mathrm{x}^{(1,2)}\| < \delta$ $(0 \leqslant \|\mathrm{x}^{(3)}\| < \infty)$ implies

$$\|\mathrm{x}^{(1,2)}(t; t_0, \mathrm{x}_0)\| < \varepsilon, \quad \forall t \geqslant t_0$$

for any $t_0 \geqslant 0$ and $\mathrm{x}_0 \in R^n$ the solution $\mathrm{x}(t; t_0, \mathrm{x}_0)$ has the property

$$\|\mathrm{x}^{(2)}(t; t_0, \mathrm{x}_0)\| = 0 \tag{2.9}$$

Here, the domain of $\mathrm{x}^{(2)}$-attraction of the point $\mathrm{x} = 0$ is the entire space.

Theorem 2. For $\mathrm{x}^{(1,2)}$-stability, uniform with respect to t_0, or asymptotic $\mathrm{x}^{(2)}$ stability in the large, of the solution $\mathrm{x} = \mathrm{x}^{(1,3)} \equiv 0$ of system (1.1), it suffices that there exist in the domain G of (1.9) an $\mathrm{x}^{(1,2)}$-positive definite function $V(t, \mathrm{x})$, admitting of an infinitesimal upper limit with respect to $\mathrm{x}^{(1,2)}$, or an infinitely large lower limit with respect to $\mathrm{x}^{(2)}$, whose total time derivative, taken with the opposite sign, i.e., $-dV/dt$, is an $\mathrm{x}^{(2)}$-positive definite function, and that we have the condition

$$\|\mathrm{X}^{(2)}\| \leqslant M > 0, \quad M \in R^1 \tag{2.10}$$

Proof. Under the conditions of Theorem 2, the conditions of Theorem 1 hold. By theorem 1, we have $\mathrm{x}^{(1,2)}$-stability, uniform with respect to t_0, and asymptotic $\mathrm{x}^{(2)}$-stability, of the solution $\mathrm{x}^{(1,3)} \equiv 0$.

Let us show that condition (2.9) holds for any $\mathrm{x}_0 \in R^n$. Since $V(t, \mathrm{x})$, in the domain G of (1.9), admits of an infinitesimal upper limit with respect to $\mathrm{x}^{(1,2)}$ and an infinitely large lower limit with respect to $\mathrm{x}^{(2)}$, then, by Definition 4 and 5. in the domain G of (1.9) we have

$$W(\mathbf{x}^{(2)}) \leqslant V(t, \mathbf{x}) \leqslant W_*(\mathbf{x}^{(1,2)}) \tag{2.11}$$

where $W(\mathbf{x}^{(2)})$, $W_*(\mathbf{x}^{(1,2)})$ are positive definite functions and $W(\mathbf{x}^{(2)}) \to \infty$, $\|\mathbf{x}^{(2)}\| \to \infty$.

On repeating the proof of asymptotic stability given in Theorem 1, and using conditions (2.10) and (2.11), we conclude that property (2.9) holds for any $\mathbf{x}_0 \in R^n$, $\forall t_0 \geqslant 0$. The theorem is proved.

Corollary. Taking $N_* = N$ in Theorems 1 and 2, we obtain corresponding theorems on the multistability of the motion with respect to all the variables. Here, in Theorem 1, $N_* = N = 3$, and in Theorem 2, $N_* = N = 2$.

3. We will use the result obtained to study the stability of the space motion of winged aircraft. We will consider the case when the aircraft, moving with fixed absolute value of the velocity, performs a manoeuvre with constant load factor. Thus, to the undisturbed motion there correspond constant values of the angles of attack α_0 and of side-slip β_0, and angular velocities of pitch ω_{z0}, yaw ω_{y0}, and rotation ω_{x0}. Their deviations from the perturbed values will be called α, β, ω_z, ω_y, ω_x respectively. The deviations of the angular velocities of side-slip, yaw, and rotation must not exceed given limits.

We consider the equations of the perturbed motion in the form /6/

$$\begin{aligned}
&\alpha^{\cdot} = \mu\omega_z - {}^1/_2 c_y{}^{\alpha}\alpha - \mu\beta\omega_x - {}^1/_2 c_y^{\delta_e}\delta_e \\
&\omega_z{}^{\cdot} = m_z{}^{\alpha}\alpha + m_z^{\omega_z}\omega_z - \mu A\omega_x\omega_y + m_z^{\delta_e}\delta_e \\
&\beta^{\cdot} = \mu\omega_y + {}^1/_2 c_z{}^{\beta}\beta + \mu\alpha\omega_x + {}^1/_2 c_z^{\delta_r}\delta_r \\
&\omega_y{}^{\cdot} = m_y{}^{\beta}\beta + m_y^{\omega_y}\omega_y + \mu B\omega_x\omega_z + m_y^{\delta_r}\delta_r \\
&\omega_x{}^{\cdot} = m_x{}^{\beta}\beta + m_x^{\omega_x}\omega_x - \mu C\omega_y\omega_z + m_x^{\delta_a}\delta_a \\
&A = \frac{J_y - J_x}{J_z} > 0, \quad B = \frac{J_z - J_x}{J_y} > 0, \quad C = \frac{J_z - J_y}{J_x} > 0
\end{aligned} \tag{3.1}$$

where μ is the aircraft relative density, $c_u{}^{\xi}$ are the coefficients of the aerodynamic forces, $m_u{}^{\xi}$ are the coefficients of the aerodynamic moments, δ_e, δ_r, δ_a are the deviations of the elevator, aileron, and rudder, and J_x, J_y, J_z are the aircraft moments of inertia with respect to the connected coordinate system.

We take the law of stabilization in the form

$$\delta_e = k_e{}^{\alpha}\alpha + k_e{}^z\omega_z, \quad \delta_r = k_r{}^{\beta}\beta + k_r{}^y\omega_y, \quad \delta_a = k_a{}^{\beta}\beta + k_a{}^x\omega_x \tag{3.2}$$

We substitute the values (3.2) into Eqs. (3.1). We use the notation

$$\begin{aligned}
&x_1 = \omega_x, \quad x_2 = \omega_y, \quad x_3 = \omega_z, \quad x_4 = \alpha, \quad x_5 = \beta \\
&a_{11} = m_x{}^{\beta} + k_a{}^{\beta} m_x^{\delta_a}, \quad a_{15} = m_x^{\omega_x} + k_a{}^x m_x^{\delta_a}, \quad a_{123} = -\mu C \\
&a_{22} = m_y{}^{\beta} + k_r{}^{\beta} m_y^{\delta_r}, \quad a_{25} = m_y^{\omega_y} + k_r{}^y m_y^{\delta_r}, \quad a_{213} = \mu B \\
&a_{33} = m_z{}^{\alpha} + k_e{}^{\alpha} m_z^{\delta_e}, \quad a_{34} = m_z^{\omega_z} + k_e{}^z m_z^{\delta_e}, \quad a_{312} = -\mu A \\
&a_{44} = {}^1/_2 (c_y{}^{\alpha} + k_e{}^{\alpha} c_y^{\delta_e}), \quad a_{43} = \mu - {}^1/_2 k_e{}^z c_y^{\delta_e}, \quad a_{415} = -\mu \\
&a_{55} = {}^1/_2 (c_z{}^{\beta} + k_r{}^{\beta} c_z^{\delta_r}), \quad a_{52} = \mu + {}^1/_2 k_r{}^y c_z^{\delta_r}, \quad a_{514} = \mu
\end{aligned} \tag{3.3}$$

Using this notation, we can write system (3.1) as

$$\begin{aligned}
x_1{}^{\cdot} &= a_{11}x_1 + a_{15}x_5 + a_{123}x_2x_3 \\
x_2{}^{\cdot} &= a_{22}x_2 + a_{25}x_5 + a_{213}x_1x_3 \\
x_3{}^{\cdot} &= a_{33}x_3 + a_{34}x_4 + a_{312}x_1x_2 \\
x_4{}^{\cdot} &= a_{43}x_3 + a_{44}x_4 + a_{415}x_1x_5 \\
x_5{}^{\cdot} &= a_{52}x_2 + a_{55}x_5 + a_{514}x_1x_4
\end{aligned} \tag{3.4}$$

We shall find the conditions connecting the coefficients of system (3.4) under which the solution of the system $\mathbf{x} = 0$ is asymptotically stable with respect to x_4, x_5, and stable with respect to x_1, x_2, x_3.

We use the corollary to Theorem 2. In our example, $N = 2$, i.e., there are two groups of variables $\{x_1, x_2, x_3\}$, $\{x_4, x_5\}$. In accordance with the notation (1.3), for system (3.4) we have

$$\mathbf{x} = \mathbf{x}^{(1,2)} = (x_1{}^{(1)}, x_2{}^{(1)}, x_3{}^{(1)}, x_4{}^{(2)}, x_5{}^{(2)})$$

We consider the Lyapunov function

$$V = {}^1/_2\,(-a_{213}a_{312}x_1^2 + 2a_{123}a_{312}x_2^2 - a_{123}a_{213}x_3^2 + x_4^2 + x_5^2) \tag{3.5}$$

which is positive definite and admits of an infinitesimal upper limit and an infinitely large lower limit with respect to the variable $x^{(1,2)}$.

In view of system (3.4) the derivative of the function (3.5) is

$$dV/dt = -a_{213}a_{312}a_{11}x_1^2 - a_{213}a_{312}a_{15}x_1x_5 + 2a_{123}a_{312}a_{22}x_2^2 + (2a_{123}a_{312}a_{25} + a_{52})\,x_2x_5 - a_{123}a_{213}a_{33}x_3^2 + (a_{43} - a_{123}a_{213}a_{34})x_3x_4 + a_{44}x_4^2 + a_{55}x_5^2 \tag{3.6}$$

By the corollary to Theorem 2, to solve our problem we have to find the conditions whereby function (3.6) is non-positive with respect to x_1, x_2, x_3 and negative definite with respect to x_4, x_5.

The method of finding these conditions is given in /7/ and is as follows. We equate the derivative dV/dt of (3.6) to the function

$$W(x) = -(c_{11}x_1 + c_{15}x_5)^2 - (c_{22}x_2 + c_{25}x_5)^2 - (c_{33}x_3 + c_{34}x_4)^2 - (c_4x_4)^2 - (c_5x_5)^2 \tag{3.7}$$

and, comparing coefficients of like terms of (3.6) and (3.7), we find the conditions for the existence of the coefficients of function (3.7) which are in fact the required conditions for the function (3.6) to be non-positive with respect to x_1, x_2, x_3 and negative definite with respect to x_4, x_5. These conditions are

$$a_{11} < 0,\quad a_{22} < 0,\quad a_{33} < 0,\quad a_{44} + \frac{(a_{43} - a_{123}a_{213}a_{34})^2}{a_{123}a_{213}a_{33}} < 0$$
$$a_{55} + \frac{a_{15}^2a_{213}a_{312}}{a_{11}} - \frac{(2a_{123}a_{312}a_{25} + a_{52})^2}{2a_{123}a_{312}a_{22}} < 0 \tag{3.8}$$

On substituting the values of the coefficients (3.3) into inequality (3.8), we obtain the sufficient conditions which solve the aircraft space manoeuvre problem.

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